

Evolution of the helicity and transversity Transverse-Momentum-Dependent parton distributions

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We examine the QCD evolution of the helicity and transversity parton distribution functions when including also their dependence on transverse momentum. Using an appropriate definition of these polarized transverse momentum distributions (TMDs), we describe their dependence on the factorization scale and rapidity cutoff, which is essential for phenomenological applications.

PACS numbers: 12.38.Bx, 13.88.+e, 12.39.St

I. INTRODUCTION

Our understanding of the partonic structure of hadrons relies on the study of parton distribution functions (PDFs) and their extensions. In the last years, particular attention has been devoted to transverse-momentum-dependent parton distribution functions (TMDs). Standard collinear PDFs are defined through collinear factorization theorems and obey the well-known DGLAP evolution equations [1–3]. TMDs are defined through transverse-momentum-dependent factorization and obey different evolution equations [4–6]. Here, for the first time we analyze these TMD evolution equations for two important distributions: the helicity and transversity TMDs.

Factorization theorems are cornerstones of our understanding of hadron structure. They describe experimentally measured cross-sections in terms of perturbatively calculable hard parts and universal structures related to nonperturbative parton dynamics, e.g., PDFs or TMDs. Factorization leads to well-defined evolution equations for the nonperturbative functions, which allows us to relate experimental measurements at different hard scales and perform global analyses of PDFs, TMDs, and the corresponding fragmentation functions.

The foundations of TMD factorization and evolution date back to Refs. [4, 7]. However, important details related to gauge invariance have been clarified only in the last decade (see, e.g., [5, 8–12]). The first proof of TMD factorization was provided by Ji, Ma, and Yuan in Refs. [5, 12] while a complete definition of TMDs and rigorous proof of factorization has been recently presented by Collins in Ref. [6] and applied in Refs. [13–16]. Another definition has been proposed in the context of Soft-Collinear Effective Theory by Echevarria, Idilbi, and Scimemi in Refs. [17, 18] (see also the closely-related work of Cherednikov and Stefanis Refs. [19, 20]). The relation between the approach of Collins and that of Echevarria, Idilbi, and Scimemi has been analyzed in Ref. [21], with the conclusion that they are essentially equivalent. In this work, we use the definition of Collins [6] and work in the framework of his TMD factorization approach.

The nonperturbative objects introduced in factorization theorems typically depend on a renormalization scale μ in the case of collinear PDFs, and a so-called rapidity cutoff ζ in the case of TMDs. Physical quantities do not depend on these artificial scales, but only on the experimentally measurable hard scale of the process (e.g., the photon invariant mass in a Drell–Yan process).

For the description of a spin-1/2 target, eight independent TMDs can be introduced (at leading twist) [22–25]. At present, there exist explicit formulas for the evolution of two of them: the unpolarized TMD, $f_1(x, \mathbf{k}_T)$, and the Sivers function, $f_{1T}^\perp(x, \mathbf{k}_T)$. Here we consider the helicity distribution, $g_1(x, \mathbf{k}_T)$ and the transversity distribution $h_1(x, \mathbf{k}_T)$. They are closely related to the collinear PDFs $g_1(x)$ and $h_1(x)$, whose collinear evolution is well known (see, e.g., Refs. [26–28]).

TMD evolution is related to transverse-momentum resummation and so called Collins-Soper-Sterman (CSS) formalism [4]. For the polarized case of interest here, studies are presented in Refs. [29]. We will clarify in which sense our results correspond to the ones presented in the resummation literature.

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II. TMD EVOLUTION

We choose a particular process, namely Semi Inclusive Deep Inelastic Scattering (SIDIS). We denote with P and S the momentum and spin vector of the hadron target, and with P_h the momentum of the detected hadron. With a single exchanged photon of momentum q , independent kinematic variables are: $Q = \sqrt{-q^2}$, $x = Q^2/2P \cdot q$, $z = P \cdot P_h/P \cdot q$, and the virtual photon's transverse momentum \mathbf{q}_T (in a hadron frame where the measured hadrons have zero transverse momentum).

Details about TMD factorization and definitions are given in Refs. [6, 13]. Here we summarize only the most important points. A SIDIS structure function in the form derived by Collins [6] reads:

$$F_{UU,T}(x, z, \mathbf{q}_T^2, Q^2) = \sum_a \mathcal{H}_{UU,T}^a(Q; \mu) \int d^2\mathbf{k}_T d^2\mathbf{p}_T f_1^a(x, \mathbf{k}_T; \mu, \zeta_F) D_1^a(z, z\mathbf{p}_T; \mu, \zeta_D) \delta^{(2)}(\mathbf{k}_T + \mathbf{q}_T - \mathbf{p}_T) + Y_{UU,T}(Q, \mathbf{q}_T) + \mathcal{O}(\Lambda/Q). \quad (1)$$

Here, $\mathcal{H}_{UU,T}$ is the hard scattering part, $f_1^a(x, \mathbf{k}_T)$ is the TMD PDF for an unpolarized quark of flavor a in an unpolarised proton, and $D_1^a(z, z\mathbf{p}_T)$ is the unpolarized fragmentation function. The formula is similar to the parton-model expression (see, e.g., [25]), except for the dependence of the functions on μ and ζ , the presence of higher-order terms in the hard scattering, and the presence of the correction term Y , which serves the purpose of correcting the expression of the structure function at large $q_T \approx Q$, where the expression in terms of TMDs is not applicable. Analogous formulas can be derived for other structure functions containing other TMD PDFs and TMD FFs, see Refs. [14, 25, 30]. Λ denotes a generic hadronic scale, e.g., Λ_{QCD} or M . Note that we work in an approximation in which light cone momentum fractions that enter in the definition of Eq. (1) and usual Bjorken variable x are equal to each other.

To correctly define the TMD PDFs and FFs in Eq. (1), it is more convenient to define them in in transverse coordinate space (\mathbf{b}_T -space) and then Fourier-transform the final result. The structure functions, as that in Eq. (1), can be written as Fourier transforms of \mathbf{b}_T -space expressions (see Sec. 2.2 of Ref. [30]).

The proper definition of TMD PDFs requires the introduction of the so-called unsubtracted TMD PDFs together with further unsubtracted functions (sometimes called “soft factors”). Both of them contain rapidity divergences that are eventually canceled in the final definition of the the TMD PDFs.

The unsubtracted TMD PDFs are defined as (dropping the flavor index)

$$\tilde{f}_1^{\text{unsub}}(x, \mathbf{b}_T; \mu, y_P - y_B) = \text{Tr} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \langle P, S | \bar{\psi}(\xi/2) W(\xi/2, \infty, n_B(y_B))^\dagger \frac{\gamma^+}{2} W(-\xi/2, \infty, n_B(y_B)) \psi(-\xi/2) | P, S \rangle_c \quad (2)$$

where $\xi^\mu = (0^+, \xi^-, \mathbf{b}_T)$, and we denote the functions with a tilde to indicate that they are defined in transverse coordinate space. The rapidity of the parent hadron is denoted by y_P . An additional parameter y_B is needed to regulate the light-cone divergences that result from using exactly light-like Wilson lines. This parameter is ultimately set to $-\infty$ in the final definition and the rapidity divergence is canceled by corresponding rapidity divergences in the soft function that we shall define below, see Ref. [6]. The subscript c indicates that only connected diagrams are included. The $W(a, b; n)$ functions represent Wilson lines (gauge links) from a to b along the direction of the four-vector n . This direction is determined by the choice of the process [10]. It is essential for calculations of evolution for T-odd functions (Sivers and Boer-Mulders functions), but does not affect the discussion of the present paper, which focuses on T-even distributions. Light cone variables are defined such as $a^\pm = (a^0 \pm a^3)/\sqrt{2}$ so that $a \cdot b = a^+b^- + a^-b^+ - \mathbf{a}_T \cdot \mathbf{b}_T$. To obtain the expressions for the helicity and transversity TMD PDFs, the Dirac structure γ^+ should be replaced by $\gamma^+\gamma_5$, and $\gamma^+\gamma^i\gamma_5$, respectively.

For the proper definition of TMD PDFs, we need also to introduce an unsubtracted soft function that corresponds to the expectation value of a Wilson loop:

$$\tilde{S}_{(0)}(\mathbf{b}_T; y_A, y_B) = \frac{1}{N_c} \langle 0 | W(\mathbf{b}_T/2, \infty; n_B)^\dagger W(\mathbf{b}_T/2, \infty; n_A) W(-\mathbf{b}_T/2, \infty; n_B) W(-\mathbf{b}_T/2, \infty; n_A)^\dagger | 0 \rangle. \quad (3)$$

In both (2) and (3), also transverse gauge links at infinity should be included [8]. However, when Feynman gauge is used their effects cancel in the final TMD PDF. Therefore we have not indicated the extra gauge links explicitly. The soft factor contains also self-interaction divergences that cancel in the final definition of TMDs.

The complete definition of the TMD PDF in \mathbf{b}_T -space, given in Refs. [6], is

$$\tilde{f}_1(x, \mathbf{b}_T; \mu, \zeta_F) = \tilde{f}_1^{\text{unsub}}(x, \mathbf{b}_T; \mu; y_P - (-\infty)) \sqrt{\frac{\tilde{S}_{(0)}(\mathbf{b}_T; +\infty, y_s)}{\tilde{S}_{(0)}(\mathbf{b}_T; +\infty, -\infty) \tilde{S}_{(0)}(\mathbf{b}_T; y_s, -\infty)}} Z_F Z_2. \quad (4)$$

Here, the “ ∞ ” arguments for the rapidity variables in the unsubtracted PDF and the soft factors are meant in the sense of a limit. All field operators are unrenormalized, and Z_F and Z_2 are the PDF and field strength renormalization factors respectively. The soft factors on the right-hand side of Eq. (4) contain rapidity arguments y_s . It is an arbitrary parameter which can be thought of as separating the plus and minus directions.

To introduce polarization effects, we denote the Bloch 3-vector for a spin-1/2 particle moving in $+z$ direction as

$$\boldsymbol{\rho} = (\boldsymbol{\rho}_T, \lambda), \quad (5)$$

where λ is the helicity and $\boldsymbol{\rho}_T$ is transverse spin, so that in massless limit one has

$$\frac{1}{2} \sum_s u(p) \bar{u}(p) = \frac{1}{2} \not{p} \left(1 - \lambda \gamma_5 - \sum_{j=1,2} \gamma_5 \rho_T^j \gamma^j \right). \quad (6)$$

If we define the following 4-vector

$$\rho_T^\mu \equiv (0^+, 0^-, \boldsymbol{\rho}_T), \quad (7)$$

then we can rewrite Eq. (6) as

$$\frac{1}{2} \sum_s u(p) \bar{u}(p) = \frac{1}{2} \not{p} \left(1 - \lambda \gamma_5 + \gamma_5 \not{\rho}_T \right), \quad (8)$$

this equation will prove useful when calculating Feynman diagrams. We also choose an appropriate Sudakov decomposition for vectors introducing two light cone vectors:

$$p \equiv (p^+, 0^-, \mathbf{0}_T), \quad (9)$$

$$n \equiv (0^+, 1^-, \mathbf{0}_T). \quad (10)$$

The Fourier transform for TMD PDFs in $D = 4 - 2\epsilon$ reads

$$\tilde{f}(x, \mathbf{b}_T) \equiv \int d^{2-2\epsilon} \mathbf{k}_T e^{-i \mathbf{b}_T \cdot \mathbf{k}_T} f(x, \mathbf{k}_T). \quad (11)$$

The inverse Fourier transform reads

$$f(x, \mathbf{k}_T) \equiv \int \frac{d^{2-2\epsilon} \mathbf{b}_T}{(2\pi)^{2-2\epsilon}} e^{i \mathbf{b}_T \cdot \mathbf{k}_T} \tilde{f}(x, \mathbf{b}_T). \quad (12)$$

In order to address the problem of TMD evolution, we have first of all to compute the TMD PDFs in a parton-target model (see Fig. 1.a). We will study distribution of a parton of type j and momentum k in a parton of type i and momentum p . At tree level (Fig. 1.a and the analogous case for gluons), the results for the TMD PDFs we want to consider are

$$f_{1i}^{\text{unsub}[0]j}(x, \mathbf{k}_T) = \delta(1-x) \delta^{(2)}(\mathbf{k}_T) \delta_i^j, \quad (13)$$

$$g_{1i}^{\text{unsub}[0]j}(x, \mathbf{k}_T, \lambda) = \delta(1-x) \delta^{(2)}(\mathbf{k}_T) \delta_i^j, \quad (14)$$

$$h_{1i}^{\text{unsub}[0]j}(x, \mathbf{k}_T, \boldsymbol{\rho}_T) = \delta(1-x) \delta^{(2)}(\mathbf{k}_T) \delta_i^j, \quad (15)$$

where f_1, g_1, h_1 denote unpolarised distribution, helicity distribution and transversity distribution respectively.

Including virtual gluon emission diagrams (see Fig. 1) leads to multiplicative corrections to Eqs. (13, 14, 15) of the form

$$f_1^{\text{unsub}[1]j}(x, \mathbf{k}_T) = f_1^{\text{unsub}[0]j}(x, \mathbf{k}_T) \cdot \mathcal{C}, \quad (16)$$

where \mathcal{C} is the appropriate result for the virtual gluon loop:

$$\mathcal{C} = -i g^2 \mu^{2\epsilon} C_F \delta_i^j \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{\text{Tr}(\not{p}(\not{p} - \not{l})\not{p}\not{p})}{4(l^2 + i\epsilon)((p-l)^2 + i\epsilon)(n \cdot (p-k-l) + i\epsilon)} \delta(p^+ - k^+) \delta^{(2)}(\mathbf{k}_T). \quad (17)$$

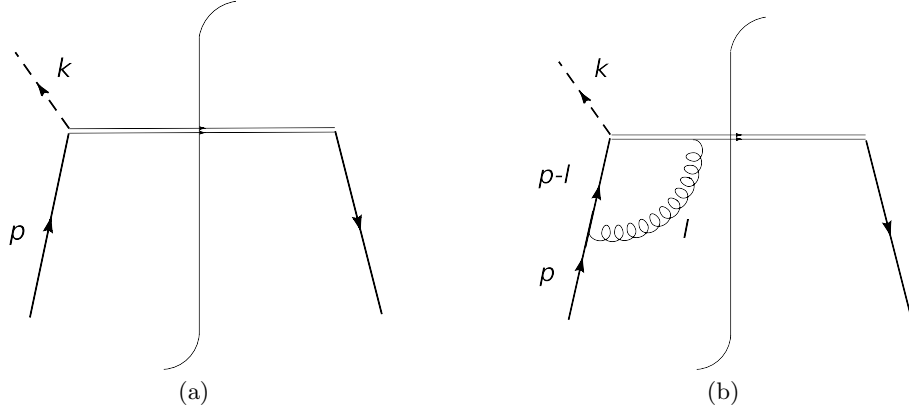


FIG. 1: Tree level diagram (a) and example of virtual gluon emission diagram (b) for the calculation of TMD PDFs in quark target model, p is the momentum of the parent quark containing quark k .

The evolution of TMDs follows from their definitions, Eqs. (4). The rapidity evolution (with respect to ζ_F) is given by the Collins–Soper (CS) equation [7]:

$$\frac{\partial \ln \tilde{f}(x, \mathbf{b}_T; \mu, \zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(\mathbf{b}_T; \mu) \quad (18)$$

where the function $\tilde{K}(\mathbf{b}_T; \mu)$ is defined as,

$$\tilde{K}(\mathbf{b}_T; \mu) = \frac{1}{2} \frac{\partial}{\partial y_s} \ln \left(\frac{\tilde{S}(\mathbf{b}_T; y_s, -\infty)}{\tilde{S}(\mathbf{b}_T; +\infty, y_s)} \right). \quad (19)$$

Note that the rapidity evolution depends only on the Soft factor, which is independent of the polarization of the quark [31]. Note that the equation contains $\tilde{S}(\mathbf{b}_T)$ rather than $\tilde{S}_{(0)}(\mathbf{b}_T)$. Thus it is important to account for the UV renormalization factors $Z_F Z_2$ in Eq. (4).

The dependence on the scale μ arises from renormalization group equations for both $\tilde{f}(x, \mathbf{b}_T; \mu, \zeta_F)$ and $\tilde{K}(\mathbf{b}_T; \mu)$. They are

$$\frac{d\tilde{K}(\mathbf{b}_T; \mu)}{d \ln \mu} = -\gamma_K(g(\mu)) \quad (20)$$

and

$$\frac{d \ln \tilde{f}(x, \mathbf{b}_T; \mu, \zeta_F)}{d \ln \mu} = \gamma_F(g(\mu); \zeta_F/\mu^2). \quad (21)$$

Here, $g(\mu)$ simply denotes the strong coupling with its dependence on the scale. The functions $\gamma_K(g(\mu))$ and $\gamma_F(g(\mu); \zeta_F/\mu^2)$ are the anomalous dimensions of $\tilde{K}(\mathbf{b}_T; \mu)$ and $\tilde{f}(x, \mathbf{b}_T; \mu, \zeta_F)$ respectively. The fact that γ_K and γ_F are independent of \mathbf{b}_T is due to the fact that UV divergence arises from virtual gluon emission diagrams only, see discussion in Ref. [32]. As mentioned earlier, those diagrams give multiplicative factors, thus the overall result on evolution does not depend on the polarization of the quark. In conclusion, the results derived in Ref. [6, 13] do not depend on the gamma matrix structure Γ used to define the specific polarized TMD, therefore they are universal for all polarization states and allow us to write down immediately the results for evolution of helicity and transversity TMD PDFs.

There is a part where polarization is important. In the regime where \mathbf{k}_T is large compared to the hadronic scale, but still small compared to the hard scale (i.e., $\Lambda \ll |\mathbf{k}_T| \ll Q$), TMDs can be calculated within a collinear factorization formalism [33–35]. This means that when \mathbf{b}_T is small but still larger than the inverse of the hard scale, i.e., $1/Q \ll |\mathbf{b}_T| \ll 1/\Lambda$, Eq. (4) can be written as the convolution of a perturbatively calculable hard scattering coefficient and an integrated PDF:

$$\tilde{f}_1^j(x, \mathbf{b}_T; \mu, \zeta_F) = \sum_{j'} \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{j/j'}(x/\hat{x}, \mathbf{b}_T; \mu, \zeta_F) f_1^{j'}(\hat{x}; \mu) + \mathcal{O}(\Lambda b_T), \quad (22)$$

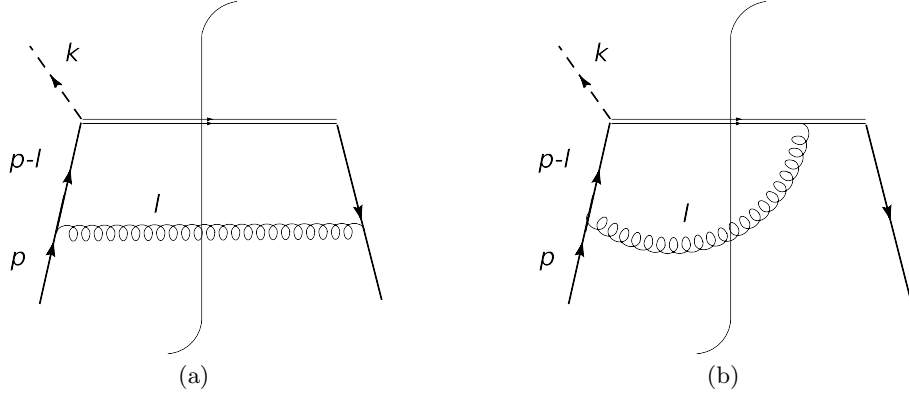


FIG. 2: Real gluon emission diagrams (a) and (b) of TMD PDF in quark target model, p is the momentum of the parent quark containing quark k . Hermitean conjugate diagrams have also to be taken into account.

the sum j' goes over all quark and antiquark q , antiquark \bar{q} flavors and gluon g . The functions $f_1(\hat{x}; \mu)$ are the ordinary integrated PDFs and the $\tilde{C}_{j/j'}(x/\hat{x}, \mathbf{b}_T; \mu, \zeta_F)$ are the hard coefficient functions. Similar expressions can be written for the helicity and transversity distribution. The hard coefficients will be different and will be denoted by $\Delta\tilde{C}$ for helicity and $\delta\tilde{C}$ for transversity. We will explicitly calculate them for helicity and transversity distribution function.

Being independent of the type of initial hadron, the computation of the hard coefficients can be performed for the parton-target case [6, 13]. We can write perturbative results for the TMD distribution at small \mathbf{b}_T up to the first order of perturbation expansion as (removing for convenience the dependence on the scales)

$$\tilde{f}_{1i}^{[1]j}(x, \mathbf{b}_T) = \tilde{C}_{j/j'}^{[1]}(x/\hat{x}, \mathbf{b}_T) \otimes f_{1i}^{[0]j'}(\hat{x}) + \tilde{C}_{j/j'}^{[0]}(x/\hat{x}, \mathbf{b}_T) \otimes f_{1i}^{[1]j'}(\hat{x}). \quad (23)$$

The symbol \otimes means the convolution from Eq. (22). The lowest order result for the hard coefficient is simply

$$\tilde{C}_{j/j'}^{[0]}(x/\hat{x}, \mathbf{b}_T) = \delta(1 - x/\hat{x})\delta_{j'j}. \quad (24)$$

Using the results for the lowest order of $f^{[0]}$ from Eqs.(13, 14, 15) integrated over \mathbf{k}_T we obtain

$$\tilde{C}_{j/i}^{[1]}(x, \mathbf{b}_T) = \tilde{f}_{1i}^{[1]j}(x, \mathbf{b}_T) - f_{1i}^{[1]j}(x). \quad (25)$$

This expression represents the recipe to compute the hard coefficients at order α_S . Analogous formulas hold for the hard coefficients $\Delta\tilde{C}$ and $\delta\tilde{C}$ of the helicity and transversity distributions.

For the TMD PDF of a quark in a quark we follow the steps of Refs. [6, 13]. To deal with eikonal propagators, we use the Feynman rules from Refs. [6, 33]. From diagrams (a) and (b) in Fig. 2 we obtain

$$\begin{aligned} f_{1i}^{[1]j}(x, \mathbf{k}_T) \Big|_a &= g^2 \mu^{2\epsilon} C_F \delta_i^j \int \frac{dk^- d^{4-2\epsilon}l}{(2\pi)^{3-2\epsilon}} \frac{\text{Tr} \left(\not{n}(\not{p} - \not{l}) \gamma_\alpha \not{n} \left(1 - \lambda \gamma_5 + \gamma_5 \not{p}_T \right) \gamma^\alpha (\not{p} - \not{l}) \right)}{4((p-l)^2 + i\varepsilon)^2} \\ &\times \delta^{(2)}(\mathbf{l}_T + \mathbf{k}_T) \delta(p^+ - l^+ - k^+) \delta(l^2), \end{aligned} \quad (26)$$

$$\begin{aligned} f_{1i}^{[1]j}(x, \mathbf{k}_T) \Big|_b &= -g^2 \mu^{2\epsilon} C_F \delta_i^j \int \frac{dk^- d^{4-2\epsilon}l}{(2\pi)^{3-2\epsilon}} \frac{\text{Tr} \left(\not{n}(\not{p} - \not{l}) \not{n} \not{p} \left(1 - \lambda \gamma_5 + \gamma_5 \not{p}_T \right) \right)}{4((p-l)^2 + i\varepsilon)(n \cdot l - i\varepsilon)} \\ &\times \delta^{(2)}(\mathbf{l}_T + \mathbf{k}_T) \delta(p^+ - l^+ - k^+) \delta(l^2). \end{aligned} \quad (27)$$

The Dirac structure \not{n} has to be replaced by $\not{n}\gamma_5$ and $\not{n}\gamma^i\gamma_5$ for the helicity and transversity distributions, respectively.

Calculations of integrals in Eqs. (26,27) go along the lines of Ref. [6]. After estimating the trace in $4 - 2\epsilon$ dimensions we evaluate the integral over dl^- first by closing the contour at infinity and using Cauchy's integral theorem. Then we evaluate the reminding dl^+ and $d^{2-2\epsilon}\mathbf{l}_T$ integrals utilizing delta functions. Finally we compute \overline{MS} counterterms by prescription from Ref. [6].

Using Eq. (25) and results of Eqs. (26,27) along with the soft subtraction factors (see Appendix A of Ref. [13]) for the TMD PDF for finding a quark of flavor j' in a quark of flavor j we find to order α_s ,

$$\begin{aligned} \tilde{C}_{j'/j}(x, \mathbf{b}_T; \mu; \zeta_F/\mu^2) = & \delta_{j'j} \delta(1-x) + \delta_{j'j} \frac{\alpha_s C_F}{\pi} \left\{ \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \left(\frac{1+x^2}{1-x} \right)_+ + \frac{1}{2}(1-x) + \right. \\ & \left. + \delta(1-x) \left[-\ln^2 \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) + \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \ln \left(\frac{\zeta_F}{\mu^2} \right) \right] \right\} + \mathcal{O}(\alpha_s^2), \quad (28) \end{aligned}$$

$$\begin{aligned} \Delta \tilde{C}_{j'/j}(x, \mathbf{b}_T; \mu; \zeta_F/\mu^2) = & \delta_{j'j} \delta(1-x) + \delta_{j'j} \frac{\alpha_s C_F}{\pi} \left\{ \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \left(\frac{1+x^2}{1-x} \right)_+ + \frac{1}{2}(1-x) + \right. \\ & \left. + \delta(1-x) \left[-\ln^2 \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) + \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \ln \left(\frac{\zeta_F}{\mu^2} \right) \right] \right\} + \mathcal{O}(\alpha_s^2), \quad (29) \end{aligned}$$

$$\begin{aligned} \delta \tilde{C}_{j'/j}(x, \mathbf{b}_T; \mu; \zeta_F/\mu^2) = & \delta_{j'j} \delta(1-x) + \delta_{j'j} \frac{\alpha_s C_F}{\pi} \left\{ \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \left(\frac{2x}{1-x} \right)_+ + \right. \\ & \left. + \delta(1-x) \left[-\ln^2 \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) + \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) \ln \left(\frac{\zeta_F}{\mu^2} \right) \right] \right\} + \mathcal{O}(\alpha_s^2). \quad (30) \end{aligned}$$

for unpolarised, helicity, and transversity TMDs respectively. The strong coupling α_s is evaluated at a scale μ , and the number of active flavors is n_f . The usual $SU(N_c)$ color factors are $C_F = (N_c^2 - 1)/(2N_c)$, $T_f = 1/2$.

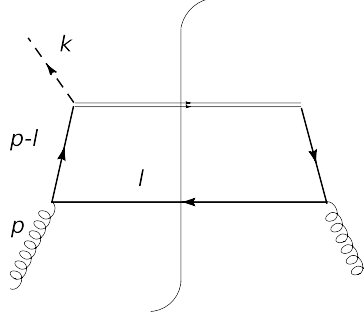


FIG. 3: Quark in a gluon diagram for TMD PDF, p is the momentum of the parent gluon containing quark k .

The calculation for quark in gluon follow the lines of Refs. [6, 13]. We utilize the diagram from Fig. 3. Note that at the order α_s there is no contribution from soft factor subtraction and all the diagrams with gluon attached to the Wilson line give zero if an appropriate choice of gluon polarization is made, see Refs. [6]. Furthermore, the diagram from Fig. 3 gives zero for transversity, since using the transversity projector $\gamma^+ \gamma^i \gamma_5$ the Dirac trace contains odd number of Dirac matrices. This result is well known in the literature.

Using Feynman rules from Ref. [6] we obtain for diagrams in Fig. 3

$$f_{1g}^{[1]j}(x, \mathbf{k}_T) = -\frac{T_f g^2 \mu^{2\epsilon}}{4(1-\epsilon)} g_{\perp}^{\mu\nu} \int \frac{dk^- d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{\text{Tr}(\not{p} \not{k} \gamma_{\mu} (\not{k} - \not{p}) \gamma_{\nu} \not{k})}{(k^2 + i\varepsilon)^2} \delta((p-k)^2), \quad (31)$$

$$g_{1g}^{[1]j}(x, \mathbf{k}_T) = -\frac{T_f g^2 \mu^{2\epsilon}}{4(1-3\epsilon)} \epsilon_{\perp}^{\mu\nu} \int \frac{dk^- d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{\text{Tr}(\not{p} \gamma_5 \not{k} \gamma_{\mu} (\not{k} - \not{p}) \gamma_{\nu} \not{k})}{(k^2 + i\varepsilon)^2} \delta((p-k)^2), \quad (32)$$

for unpolarised and helicity distributions accordingly. Here

$$g_{\perp}^{\mu\nu} \equiv -g^{\mu\nu} + \frac{1}{p^+} (p^{\mu} n^{\nu} + n^{\mu} p^{\nu}), \quad (33)$$

$$\epsilon_{\perp}^{\mu\nu} \equiv \frac{1}{p^+} \epsilon^{\alpha\beta\mu\nu} p_{\alpha} n_{\beta}. \quad (34)$$

In conclusion, for the gluon contributions we obtain

$$\tilde{C}_{g/j}(x, \mathbf{b}_T; \mu, \zeta_F/\mu^2) = \frac{\alpha_s T_f}{\pi} \left\{ \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) (x^2 + (1-x)^2) + x(1-x) \right\} + \mathcal{O}(\alpha_s^2). \quad (35)$$

$$\Delta \tilde{C}_{g/j}(x, \mathbf{b}_T; \mu, \zeta_F/\mu^2) = \frac{\alpha_s T_f}{\pi} \left\{ \ln \left(\frac{2e^{-\gamma_E}}{\mu b_T} \right) (2x-1) + (1-x) \right\} + \mathcal{O}(\alpha_s^2). \quad (36)$$

$$\delta \tilde{C}_{g/j}(x, \mathbf{b}_T; \mu, \zeta_F/\mu^2) = 0 + \mathcal{O}(\alpha_s^2). \quad (37)$$

for unpolarised, helicity, and transversity TMDs, respectively. Note that there are no contributions from Soft factor at this order to quark in a gluon coefficient functions.

Eqs. (29, 30) and Eqs. (36, 37) represent the original results of this paper. They allow us to write expressions for the helicity and transversity TMDs that fulfill TMD evolution equations, Eqs. (18, 20, 21), and have a behavior at high transverse momentum that matches perturbative calculations. The solution for a TMD for flavor i can be written in a compact way as (see Refs. [6, 13])

$$\tilde{f}_1^f(x, b_T; \mu, \zeta_F) = \sum_i (\tilde{C}_{f/i} \otimes f_1^i)(x, b_*; \mu_b) e^{\tilde{S}(b_*; \mu_b, \mu, \zeta_F)} e^{g_K(b_T) \ln \frac{\sqrt{\zeta_F}}{\sqrt{\zeta_{f0}}}} \hat{f}_{\text{NP}}^q(x, b_T) \quad (38)$$

Analogous formulas hold for the helicity, g_1 , or transversity, h_1 , distributions. The sum goes over all quark and antiquark flavors and include also gluon, $j = q, \bar{q}, g$. Appropriate coefficient functions should be used in each case from Eqs. (28, 29, 30), \otimes denotes the convolution in longitudinal momentum fractions of Eq. (22). The scale μ_b is chosen appropriately to ensure the optimal convergence of perturbative series. The function \hat{f}_{NP} denotes the nonperturbative part of the TMD and has to be fitted to experimental data. In the literature, it is usually parametrized as a Gaussian, although there is no fundamental reason for this choice.

In order to be able to use Eq. (22) also at large b_T , the so-called b_* prescription can be introduced. The function b_* serves the purpose of freezing the value of b_T , preventing it from becoming larger than a certain value and avoid regions where the perturbative running coupling α_s becomes divergent (at very large values of b_T , or equivalently, at very small transverse momentum). A common choice is to set

$$b_* \equiv \frac{b_T}{\sqrt{1 + b_T^2/b_{\text{max}}^2}}, \quad (39)$$

but other functional forms can be explored, as well as different prescriptions (e.g., the complex- b prescription of Ref. [36, 37]). Any change in the prescription should also be combined with a change of the nonperturbative function \hat{f}_{NP} .

The perturbatively calculable function $\tilde{S}(b_*; \mu_b, \mu, \zeta_F)$ reads

$$\tilde{S}(b_*; \mu_b, \mu, \zeta_F) = \ln \frac{\sqrt{\zeta_F}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta_F}}{\mu'} \gamma_K(g(\mu')) \right], \quad (40)$$

and expressions for \tilde{K} , γ_F , and γ_K at order α_s can be found in Appendix B of Ref. [13].

We adopt also these other choices ¹

$$\mu = Q, \quad \mu_b = 2e^{-\gamma_E}/b_* \equiv b_0/b_*, \quad \zeta_F = Q^2, \quad \zeta_{F0} = Q_0^2. \quad (41)$$

Different options can be explored in order to test the sensitivity of the final results to the scale choice. Note that $\tilde{K}(b_*; \mu_b) = 0$ at this order with this choice.

Using Eq. (41) we find to order α_s for Eqs.(28,29,30),

$$\tilde{C}_{j'/j}(x, b_*; \mu_b) = \delta_{j'j} \delta(1-x) + \delta_{j'j} \frac{\alpha_s C_F}{2\pi} (1-x) + \mathcal{O}(\alpha_s^2), \quad (42)$$

$$\Delta \tilde{C}_{j'/j}(x, b_*; \mu_b) = \delta_{j'j} \delta(1-x) + \delta_{j'j} \frac{\alpha_s C_F}{2\pi} (1-x) + \mathcal{O}(\alpha_s^2), \quad (43)$$

$$\delta \tilde{C}_{j'/j}(x, b_*; \mu_b) = \delta_{j'j} \delta(1-x) + \mathcal{O}(\alpha_s^2). \quad (44)$$

¹ Note that parameter b_0 is dimensionless, so that b_0/b_* has the dimensions of energy, GeV.

and for Eqs.(35,36,37),

$$\tilde{C}_{g/j}(x, b_*; \mu_b) = \frac{\alpha_s T_f}{\pi} x(1-x) + \mathcal{O}(\alpha_s^2), \quad (45)$$

$$\Delta \tilde{C}_{g/j}(x, b_*; \mu_b) = \frac{\alpha_s T_f}{\pi} (1-x) + \mathcal{O}(\alpha_s^2), \quad (46)$$

$$\delta \tilde{C}_{g/j}(x, b_*; \mu_b) = \mathcal{O}(\alpha_s^2). \quad (47)$$

for unpolarised, helicity, and transversity TMDs, respectively. One can see that $\tilde{C}_{j'/j} = \Delta \tilde{C}_{j'/j}$ and the difference between $\tilde{C}_{j'/j}$ and $\delta \tilde{C}_{j'/j}$ is the absence of α_s contribution.

At this point, a discussion on the comparison with the CSS literature is in order. The restructuring of the formalism in terms of TMD definitions makes it non-trivial to map them onto the classic CSS components. However, we can check that the final result for the structure functions match. For unpolarized DIS, results in the CSS formalism are presented, e.g., in Ref. [38]. Polarized DIS has been discussed in Ref. [29]. For instance, we can compare the results for the unpolarized structure function of Eq. (1). We need the expression of the hard scattering, which has been reported in Ref. [39]

$$\mathcal{H}_{UU,T}^a(Q; \mu) = e_a^2 \left(1 + \frac{C_F \alpha_s}{\pi} \left[\frac{3}{2} \ln \left(\frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left(\frac{Q^2}{\mu^2} \right) - 4 \right] \right). \quad (48)$$

If we insert the expression for the unpolarized TMD, Eq. (38), the analogous expression for the fragmentation function, Eq. (31) of Ref. [13], and the hard scattering, Eq. (48), in the formula of the structure function, Eq. (1) and we adopt the choices of Eq. (41), we recover the standard CSS results for DIS at order α_s (see, e.g., Eq. (45) of Ref. [38], or Eq. (32) of Ref. [29]). Other structure functions for polarized DIS require the study of different TMD PDFs and FFs, together with a careful assesment of the possibility to match the low and high transverse momentum results (see, e.g., Ref. [35, 40, 41]). Since here we are concerned only with helicity and transversity TMDs, results can be compared with those of Ref. [29].

Our results can be compared also to the Drell–Yan case. The unpolarized case has been discussed in many papers, e.g., Refs. [4, 42–45]). In this case, however, the hard scattering of Eq. (48) has an extra $\pi^2/2$ in the square brackets, as in Eq. (6) of Ref. [39]). Our results can be compared to those for doubly longitudinally and transversely polarized Drell–Yann [46, 47].

Apart from the full structure function, we can focus our attention on the coefficient functions: our results expressed in Eqs. (42, 43, 44) and in Eqs. (45, 46, 47) correspond to Eq. (37, 38, 41, 42, 43) of Ref. [29]. Differences occur only for the terms with $\delta(1-x)$ and can be ascribed to the contribution from the hard scattering. These contributions turn out to be the same for all three cases, which indicates that the hard scattering is the same for the relevant structure functions. This has been already observed, albeit with a different definition of the hard scattering, in Refs. [12, 41]. Since the hard scattering is different in different processes, in the CSS formalism the coefficients are different in DIS and Drell–Yan scattering. This can be checked by comparing, e.g., Eq. (38, 42) of Ref. [29] with the coefficient Δc_q in Eq. (15) of Ref. [46] and Eq. (38, 43) of Ref. [29] with the α_s part of Eq. (8) of Ref. [47]. The difference between these results is due to the factor $\pi^2/2$ that we discussed previously.

III. PHENOMENOLOGY

Let us finally present results of TMD evolution using a test function at initial scale $Q_0 = 3.2$ GeV. We choose the collinear functions in the following form:

$$xu_0(x) = xd_0(x) \equiv x^{0.5}(1-x)^{0.5}, \quad x\bar{u}_0(x) = x\bar{d}_0(x) \equiv 0, \quad xg_0(x) \equiv x^{0.5}(1-x)^{0.5}. \quad (49)$$

Note that the choice is arbitrary as our goal is just to demonstrate the results of evolution. These initial distributions will be assumed the same for unpolarised (f_1), helicity (g_1), and transversity (h_1) distributions. Results with realistic initial functions will be presented elsewhere.

Evolution in b_T is identical for all three functions, see Eq. (38). The convolution in Eq. (38) takes three different forms. For the unpolarised distribution, using Eqs. (42, 45) we find

$$\sum_j (C_{j/i} \otimes f_1^j)(x, b_*; \mu_b) = f_1(x, \mu_b) + \frac{\alpha_s C_F}{2\pi} \int_x^1 \frac{d\hat{x}}{\hat{x}} \left(1 - \frac{x}{\hat{x}} \right) f_1(\hat{x}, \mu_b) + \frac{\alpha_s T_f}{\pi} \int_x^1 \frac{d\hat{x}}{\hat{x}} \frac{x}{\hat{x}} \left(1 - \frac{x}{\hat{x}} \right) g(\hat{x}, \mu_b), \quad (50)$$

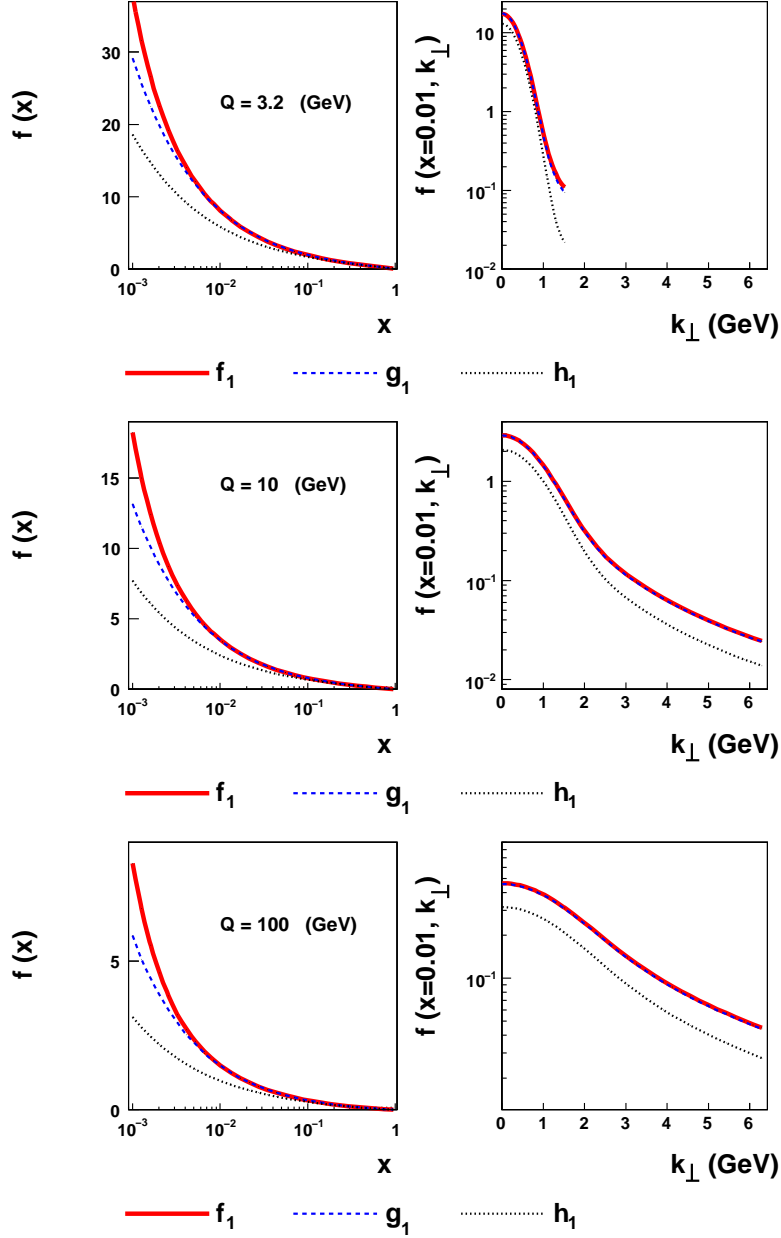


FIG. 4: Evolution of $f^q(x, k_T; Q, Q^2)$ and $f^q(x; Q, Q^2)$ at two different scales $Q = 3.2$ GeV (two upper plots), 10 GeV (two middle plots), and 100 GeV (two bottom plots). Solid line corresponds to unpolarised evolution, dashed line corresponds to helicity evolution, and dotted line corresponds to transversity evolution.

where $g(\hat{x}, \mu_b)$ corresponds to the gluon distribution.

For the helicity distribution, using Eqs. (43, 46) we obtain

$$\sum_j (\Delta C_{j/i} \otimes g_1^j)(x, b_*; \mu_b) = g_1(x, \mu_b) + \frac{\alpha_s C_F}{2\pi} \int_x^1 \frac{d\hat{x}}{\hat{x}} \left(1 - \frac{x}{\hat{x}}\right) g_1(\hat{x}, \mu_b) + \frac{\alpha_s T_f}{\pi} \int_x^1 \frac{d\hat{x}}{\hat{x}} \left(1 - \frac{x}{\hat{x}}\right) \Delta g(\hat{x}, \mu_b). \quad (51)$$

Finally, for the transversity distribution, using Eqs. (44, 47) we obtain

$$\sum_j (\delta C_{j/i} \otimes h_1^j)(x, b_*; \mu_b) = h_1(x, \mu_b). \quad (52)$$

Note that, as is well known for CSS resummation, at this order TMD evolution does not have mixing of different

flavors and no mixing with gluon TMD, even though collinear PDFs mix as can be seen from Eqs. (50,51,52). This means also that the gluon TMD cannot be studied via scaling violations at least at this order.

We perform DGLAP evolution for the collinear functions $f_1(x, \mu_b)$, $g_1(x, \mu_b)$, $h_1(x, \mu_b)$ using the HOPPET evolution package [48].

The choices of non-perturbative functions that enter in Eq. (38) are the following (we use again the same function for all three polarization cases just for illustration purposes):

$$\hat{f}_{\text{NP}}^q(x, b_T) = \exp\left(-\frac{b_T^2 \langle k_T^2 \rangle}{4}\right), \quad g_K(b_T) = -g b_T^2, \quad (53)$$

where $\langle k_T^2 \rangle = 0.25 \text{ (GeV}^2\text{)}$, $g = 0.2 \text{ (GeV}^2\text{)}$. We also choose $b_{\text{max}} = 1.5 \text{ (GeV}^{-1}\text{)}$.

The choice of \hat{f}_{NP}^q corresponds to non-perturbative functions used in analysis of TMD functions at tree-level by the Torino-Cagliari-JLab group [49, 50]. The choice of $g_K(b_T)$ is motivated by the so-called LBNY fit of Drell-Yan cross-sections using CSS resummation formalism [51].

We will show the evolution of the TMD functions, as well as their integral over k_T up to the value of Q , which we conventionally referred to as their 0th k_T -moment:

$$f^q(x, k_T; \mu, \zeta_F), \quad f^q(x; \mu, \zeta_F) \equiv 2\pi \int_0^\mu k_T dk_T f_{q/P}(x, k_T; \mu, \zeta_F). \quad (54)$$

Note that this 0th k_T -moment of a TMD functions $f^q(x; \mu, \zeta_F)$ should not be confused with collinear PDF $f^q(x; \mu)$.

In Fig. (4) we show results of the evolution of $f^q(x, k_T; Q, Q^2)$ and $f^q(x; Q, Q^2)$ at three different scales $Q = 3.2, 10$ and 100 GeV , for the unpolarized, helicity and transversity distributions. As one can see from Fig. (4), after evolution the three functions become wider and are still very similar. The only appreciable differences are at higher transverse momentum. The 0th k_T -moments after evolution show some differences at low x . Compare our results with results of collinear evolution of g_1 and h_1 presented, for example in Ref. [52]. As in the case of collinear evolution, also in TMD evolution h_1 becomes smaller than g_1 under evolution and the difference grows with Q . The reason is the absence of α_s^1 contributions to coefficient functions of transversity.

We also note that TMD functions at initial scale are very much similar to TMD functions parametrized at tree level [49, 50] which thus justify extraction of those functions at tree level. Observables at different characteristic scales however should be described using TMD evolution.

One can also observe that the so called Soffer bound [53]: $|h_1(x, Q^2)| \leq \frac{1}{2} (f_1(x, Q^2) + g_1(x, Q^2))$, is also satisfied for TMD distributions $f^q(x, k_T; Q, Q^2)$ and $f^q(x; Q, Q^2)$ numerically. Let us remind that Soffer bound for collinear densities was shown to be preserved at LO accuracy in Ref. [26] and at NLO accuracy in Ref. [27]. We set aside the discussion of Soffer bound for TMD functions for a separate publication.

IV. CONCLUSIONS

In this paper we calculated the evolution of the transverse-momentum-dependent (TMD) helicity and transversity distribution functions. We adopted the definition of TMD PDFs as given by Collins in Ref. [6]. We provided explicit formulas for all coefficient functions at α_s . The results of this paper can be readily used in TMD phenomenology.

As an illustration, we calculated the unpolarized, helicity and transversity TMD distributions at different scales, starting from the same initial conditions. The final results are very similar. Their 0th k_T -moments differ at low x . We observed that if started from equal initial conditions, helicity TMD distribution g_1 becomes smaller than unpolarised f_1 distribution and transversity h_1 becomes smaller than helicity g_1 TMD.

Acknowledgments

We would like to thank Ted Rogers for multiple discussions, help and encouragement during writing of this paper. Authored by a Jefferson Science Associate, LLC under U.S. DOE Contract No. DE-AC05-06OR23177. A. Bacchetta is partially supported by the Italian MIUR through the PRIN 2008EKLACK, and by the European Community through the Research Infrastructure Integrating Activity ‘‘HadronPhysics2’’ (Grant Agreement n. 227431) under the 7th Framework Programme.

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